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Hartly Ross Type Unbiased Estimator in Stratified Ranked Set Sampling Using Two Auxiliary Variables

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ABSTRACT

Ranked set sampling has gained much attention of the research in the recent past due to its enhanced efficiency. In this article, we proposed Hartly-Ross type unbiased estimator of the finite population mean using two auxiliary variables in stratified ranked set sampling (SRSS). The variance of the proposed unbiased estimator is derived up to first order of approximation. Comparison among the proposed and competitor estimators are made both theoretically and through rigorous simulation study. It is observed that the newly suggested Hartly-Ross type estimator is more efficient as compared to all the considered competitor estimators under SRSS design.

Keywords: Stratified Ranked set sampling, Hartly Ross estimator, efficiency.

INTRODUCTION

When the survey variable is costly and time consuming then Ranked set sampling (RSS), proposed by McIntyre (1952), proved to be more effective in reducing cost and increasing efficiency. In RSS, the variable can be ranked easily at no or lower cost. Takahasi and Wakimoto (1968) proved the mathematical theory that the sample mean under RSS is an unbiased estimator of the finite population mean and is more efficient than the sample mean estimator under simple random sampling (SRS). Stokes (1977) suggested ranking of elements on the basis of auxiliary variable instead of personal judgment.

Hartley and Ross (1954) were the first to propose an unbiased ratio estimator for finite population mean in SRS. Later, Pascual (1961) proposed an unbiased ratio type estimator in stratified random sampling. Singh et al. (2014) and Kadilar and Cekim (2014) suggested Hartley-Ross (HR) type unbiased estimators of the finite population mean using auxiliary information of population parameters in SRS. Khan and Shabbir (2016a) proposed a class of HR type unbiased estimators using RSS scheme. Khan and Shabbir (2016b) suggested several HR type unbiased estimators using known information of population parameters of auxiliary variable under RSS and SRSS design. Khan et al (2017) proposed several unbiased ratio type estimators in SRSS when population mean for the auxiliary variable is known.

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Keeping in view the vitality of HR estimator and usage of auxiliary information, we propose HR type unbiased estimator of the finite population mean using two auxiliary variables under SRSS scheme. Variance expression up to first order approximation has been derived. The newly proposed estimator is then compared with some existing mean estimators both theoretically and numerically.

SAMPLING SCHEME

In ranked set sampling, m independent random samples each of size m are chosen and the elements in each sample are selected with equal probability and without replacement from a finite population of size N.The elements of each random sample are then ranked with respect to the characteristic of the study variable or auxiliary variable. Let Y be the study variable and X and Z be the two auxiliary variables having some correlation with the study variable. The randomly selected m^2 tri-variate sample elements from the population are allocated into m sets, each of size m. Each sample is ranked with respect to one of the auxiliary variables X or Z. In this study, ranking is done on the basis of auxiliary variable X. An actual measurement from the first sample is then taken on the elements with the smallest rank of X, together with variables Y and Z associated with smallest rank of X. From second sample of size m, the variables Y and Z associated with the second smallest rank of X are measured. By this way, this procedure is continued until the Y and Z values associated with the highest rank of X are measured from the m^th sample. This completes one cycle of the sampling. The process is repeated r time to obtain the desired sample of size $n = mr$ elements.Thus, in an RSS scheme, a total of m^2r elements have been drawn from the population from which only mr of them are selected for analysis.

To estimate population, mean ($\overline{Y}$) in SRSS, the procedure is outlined as:

1. **Step 1**: Select m_h^2 tri-variate sample units randomly from the h^{th} stratum of the population.
2. **Step 2**: Arrange these selected units randomly into m_h sets, each of size m_h.
3. **Step 3**: The procedure of ranked set sampling (RSS) is then applied on each of the sets to obtain the m_h sets of ranked set samples, each of size m_h. Here, ranking is done with respect to the auxiliary variable X_h. These ranked set samples collectively form mh sets, each of size m_h units.
4. **Step 4**: Repeat the above steps r times for each stratum to get the desired sample of size $n_h = m_h r$.

Under S_tRSS scheme, the usual sample mean estimator ($\overline{y}\ SRss$) is given by;

$$\overline{\boldsymbol{y}}_{\boldsymbol{1}}^{(\boldsymbol{u})} = \sum_{\boldsymbol{h}=\boldsymbol{1}}^{\boldsymbol{L}} W_h\, \overline{y}h[rss], \qquad \textbf{(1)}$$

Where,$\overline{y}h[rss]$= $(1/m_h r) \sum_{j=1}^{r} \sum_{i=1}^{m_h} y_h[i: m_h]j\ and P_{h= Nh} / N$is the known stratum weight.

The variance of $\overline{y}$SRSS is given by;

$$V\left(\overline{y}_1^{(u)}\right) = \sum_{h=1}^{L} P_h^2\, \overline{Y}_h^2 \left(\gamma_h C_{yh}^2 - W_{yh}^2\right). \qquad (2)$$

Khan and Shabbir (2016b) suggested the following Hartly-Ross unbiased estimator in SRSS

$$\overline{y}_2^{(u)} = \sum_{h=1}^{L} P_h \left[\left(\overline{r}_{h[rss]} + \frac{b_h(\overline{X}_h - \overline{x}_{h[rss]})}{\overline{x}_{h(rss)}}\right)\overline{X}_h + \frac{\mathrm{n_h(Nh-1)}}{\mathrm{N}_h(\mathrm{nh}-1)}\left(\overline{y}_h[rss] - \overline{r}_{h[rss]}\overline{x}_{h[rss]}\right)\right],$$
(3)

and its variance is given by

$$V(\overline{y}_2^{(u)}) = \sum_{h=1}^{L} P_h^2 \left[\overline{Y}_h^2\left(\gamma_h C_{yh}^2 - W_{yh}^2\right) + \left(\beta_h + \overline{R}_h\right)^2 \overline{X}^2{}_h\left(\gamma_h C_{xh}^2 - W_{xh}^2\right)\right]$$
$$-2(1+\overline{R}_h)\,\overline{X}_h\overline{Y}_h\left(\gamma_h C_{yh\,xh}^2 - W_{yh\,xh}^2\right). \quad (4)$$

Alternatively, Khan et al (2017) suggested a more improved Hartly-Ross type unbiased estimator in RSS. This estimator along with its approximate variance is appended below.

$$\overline{y}_3^{(u)} = \sum_{h=1}^{L} P_h \left[\overline{r}_{h[i:m_h]}\overline{X}_h + \frac{\mathrm{n_h(Nh-1)}}{\mathrm{N}_h(\mathrm{nh}-1)}\left(\overline{y}_{h[i:m_h]} - \overline{r}_{h[i:m_h]}\overline{x}_{h[i:m_h]}\right)\right], \quad (5)$$

$$V(\overline{y}_3^{(u)}) = \sum_{h=1}^{L} P_h^2 \left[\overline{Y}_h^2\left(\gamma_h C_{yh}^2 - W_{yh[i:m_h]}^2\right) + \overline{X}^2{}_h\,\overline{R}^2{}_h\left(\gamma_h C_{xh}^2 - W_{xh[i:m_h]}^2\right)\right]$$
$$-2\overline{R}_h\overline{X}_h\overline{Y}_h\left(\gamma_h C_{yh\,xh}^2 - W_{yh\,xh}^2\right). \quad (6)$$

Proposed Hartly-Ross Type Unbiased Estimator in SRSS

Inspired by the work of Khan and Shabbir (2017), we suggest the following Hartly-Ross type unbiased estimator in SRSS.

$$\overline{y}_{SRSS} = \sum_{h=1}^{L} P_h \left[\overline{r}_{h[rss]}\overline{g}_{h\,[rss\ ]}\overline{Z}_h\right] \quad (7)$$

Where, $\overline{r}_{h[rss]} = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m_h} r_{[i:m_h]j}}{m_h r}$, $r[i:m_h]j, = \frac{y_{[i:m_h]j}}{x_{[i:m_h]j}}$ $\overline{g}_{h\,[rss\ ]} = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m_h} g[i:m_h]j}{m_h r}$

$$g_{[i:mh]j} = \frac{x_{[\ i:mh]i}}{z_{[\ i:mh]i}} \quad \overline{Z}_h = \frac{\sum_{h=1}^{L} z_i}{N_h}$$

Now

$$E(\overline{y}_{SRSS} =)E\left[\sum_{h=1}^{L} P_h \left[\overline{r}_{h[rss]}\overline{g}_{h\,[rss\ ]}\overline{Z}_h\right]\right]$$

$$= \sum_{h=1}^{L} P_h \left[\left(\frac{N_h - 1}{n_h N_h}\right)\overline{Z}_h S_{h[i:mh]} + \overline{R}_h\overline{G}_h\overline{Z}_h\right]$$

Where $\overline{R}_h = E(\overline{r}_{h[rss]}), \overline{G}_h = E\left(\overline{g}_{h[rss]}\right), and\ S_{rg[i:mh]} = \frac{\sum_{j=1}^{N_h} g_{[i:mh]j}\,(r_{[i:mh]j} - \overline{R}_h}{N_h-1}$

$$\mathrm{B}(\overline{y}_{SRSS} = E(\overline{y}_{StRSS})) - \overline{Y}$$

$= \sum_{h=1}^{L} P_h \left[\frac{(N_h-1)}{n_h N_h}\overline{Z}_h S_{rg[i:mh]} - \left(\overline{Y}_h - \overline{R}_h\overline{G}_h\overline{Z}_h\right)\right]$

$= \sum_{h=1}^{L} P_h \left[\frac{(N_h-1)}{n_h N_h}\overline{Z}_h S_{rg[i:mh]} - \overline{Y}_h + \overline{R}_h\overline{G}_h\overline{Z}_h - E\left(r_{[i:mh]}g_{[i:mh]}\right)\overline{Z}_h + E\left(r_{[i:mh]}g_{[i:mh]}\right)\overline{Z}_h\right]$

$=\sum_{h=1}^{L} P_h \left[\frac{(N_h-1)}{n_h N_h}\overline{Z}_h S_{rg[i:mh]} - \overline{Y}_h + E\left(r_{[i:mh]}g_{[i:mh]}\right)\overline{Z}_h - E\left(r_{[i:mh]}g_{[i:mh]}\right)\overline{Z} + E\left(r_{[i:mh]}g_{[i:mh]}\right)\overline{Z}_h\right]$

$= \sum_{h=1}^{L} P_h \left[\frac{(N_h-1)}{n_h N_h}\overline{Z}_h S_{rg[i:mh]} - \overline{Y}_h - \frac{(N_h-1)}{N_h}\overline{Z}_h S_{rg[i:mh]} + E\left(r_{[i:mh]}g_{[i:mh]}\right)\overline{Z}\right]$

$= \sum_{h=1}^{L} P_h \left[\frac{(N_h-1)}{n_h N_h}\overline{Z}_h S_{rg[i:mh]} - \frac{(N_h-1)}{N_h}\overline{Z}_h S_{rg[i:mh]} + E\left(r_{[i:mh]}g_{[i:mh]}\right)\overline{Z}\right]$

$=\sum_{h=1}^{L} P_h \left[\frac{(N_h-1)}{n_h N_h}\overline{Z}_h S_{rg[i:mh]} - \frac{(N_h-1)}{N_h}\overline{Z}_h S_{rg[i:mh]} + E\left(\frac{y_{[i:mh]}x_{[i:mh]}}{x_{[i:mh]}z_{[i:mh]}}\right)E\left(z_{[i:mh]}\right) - E\left(\frac{y_{[i:mh]}}{z_{[i:mh]}}z_{[i:mh]}\right)\right]$

$= \sum_{h=1}^{L} P_h \left[\frac{(N_h-1)}{n_h N_h}\overline{Z}_h S_{rg[i:mh]} - \frac{(N_h-1)}{N_h}\overline{Z}_h S_{rg[i:mh]} + E\left(h_{[i:mh]}z_{[i:mh]}\right) - E\left(h_{[i:mh]}z_{[i:mh]}\right)\right]$

$$\sum_{h=1}^{L} P_h \left[\frac{(N_h-1)}{n_h N_h}\overline{Z}_h S_{rg[i:mh]} - \frac{(N_h-1)}{N_h}\overline{Z}_h S_{rg[i:mh]} - \frac{(N_h-1)}{N_h} S_{hz[i:mh]}\right]$$

Or

$$B(\overline{y}_{SRSS}) = -\sum_{h=1}^{L} P_h \left[\frac{(N_h-1)}{N_h}\left(\frac{(n_h-1)}{N_h}\overline{Z}_h S_{rg[i:mh]} + S_{hz[i:mh]}\right)\right] \quad (8)$$

Where $S_{hz[i:mh]} = \frac{1}{N_h-1}\sum_{j=1}^{N_h} h_{[i:mh]j}\left(z_{[i:mh]j} - \overline{Z}_h\right)$ *and* $h_{[i:mh]j} = \frac{y_{[i:mh]j}}{z_{[i:mh]j}}$

Theorem 1: An unbiased estimator of

$S_{rh[i:mh]j\;gh(i:mh)} = \dfrac{1}{N_h}\sum_{j=1}^{N_h}\left(r_{h[i:mh]j} - \overline{R}_h\right)\left(g_{h[i:mh]j} - \overline{G}_h\right)$ *is given by*

$$S_{rh[i:mh]j\;gh(i:mh)} = \frac{n_h}{n_h-1}\left(\overline{h}_{h[i:mh]} - \overline{r}_{h[i:mh]}\overline{g}_{h[i:mh]}\right).$$

Proof

We have to prove that $E\left(S_{rh[i:mh]j\;gh(i:mh)}\right) = S_{rh[i:mh]j\;gh(i:mh)}$.$For\ fixed\ i, j =$ $1,2, \ldots \ldots, n_h, r_{rh[i:mh]j}\ and\ gh(i:mh)\ j\ are\ simple\ random\ samples\ of\ size\ \ n_h$

E $\left(S_{rh[i:mh]j\;gh(i:mh)}\right) = E\ \left[\frac{n_h}{n_h-1}\left(\overline{h}_{h[i:mh]} - \overline{r}_{h[i:mh]}\overline{g}_{h[i:mh]}\right)\right],$

$= E\ \left[\frac{1}{n_h-1}\sum_{j=1}^{nh}\left(\left(r_{h[i:mh]j} - \overline{r}_{h[i:mh]}\right)\left(g_{h[i:mh]j} - \overline{g}_{h[i:mh]}\right)\right)\right],$

$= \frac{1}{n_h-1}E\ \left[\sum_{j=1}^{nh}\left(r_{h[i:mh]j}g_{h[i:mh]j} - n_h\,\overline{r}_{h[i:mh]j}\overline{g}_{h[i:mh]}\right)\right]$

$= \frac{1}{n_h-1}\left[\sum_{j=1}^{nh}\left(E\left(r_{h[i:mh]j}g_{h[i:mh]j}\right) - n_h E\left(\overline{r}_{h[i:mh]}\overline{g}_{h[i:mh]}\right)\right)\right],$

$=\frac{1}{n_h-1}\left[\frac{n_h}{N_h}\sum_{j=1}^{nh} r_{h[i:mh]j}g_{h[i:mh]j} - n_h\left(Cov\left(\overline{r}_{h[i:mh]}, \overline{g}_{h[i:mh]}\right) + \overline{R}_h\overline{G}_h\right)\right]$

$= \frac{n_h}{n_h-1}\left[\frac{1}{N_h}\sum_{j=1}^{N_h} r_{[i:mh]j}g_{[i:mh]j} - \overline{RG} - \frac{S_{r[i:mh]g[i:mh]}}{n_h}\right]$

$= S_{rh[i:mh]gh[i:mh]}.$

Theorem 2: An unbiased estimator of

$$S_{rh[i:mh]zh[i:mh]} = \frac{1}{N_h}\frac{1}{n_h-1}\sum_{j=1}^{N_h}\left(\left(h_{h[i:mh]j} - \overline{H}_h\right)\left(z_{h[i:mh]j} - \overline{Z}_h\right)\right)\ is\ given\ by$$

$$S_{rh[i:mh]zh[i:mh]} = \frac{n_h}{n_h-1}\left(\overline{y}_{h[i:mh]} - \overline{h}_{h[i:mh]}\overline{z}_{h[i:mh]}\right).$$

We know to prove that

$$E\left(Sh_{h[i:mh]zh[i:mh]}\right) = Sh_{h[i:mh]zh[i:mh]}$$

For fixed i, j = 1, 2, …., $n_h, h_{h[i:mh]j}$ $and\ z_{h[i:mh]j}$ are simple random samples of size n_h.

$E\left(Sh_{h[i:mh]zh[i:mh]}\right) = E\left[\frac{n_h}{n_h-1}\left(\overline{y}_{h[i:mh]} - \overline{h}_{h[i:mh]}\overline{z}_{h[i:mh]}\right)\right],$

$= E\left[\frac{n_h}{n_h-1}\sum_{j=1}^{n_j}\left(h_{h[i:mh]} - \overline{h}_{h[i:mh]}\right)\left(z_{h[i:mh]} - \overline{Z}_{h[i:mh]}\right)\right],$

$= \frac{1}{n_h-1}E\left[\sum_{j=1}^{n_j} h_{h[i:mh]}z_{h[i:mh]j} - n_h\overline{h}_{h[i:mh]}\overline{Z}_{h[i:mh]}\right],$

$= \frac{1}{n_h - 1}\left[\sum_{j=1}^{n_j} E\left(h_{h[i:mh]} z_{h[i:mh]j}\right) - n_h E\left(\overline{h}_{h[i:mh]} \overline{Z}_{h[i:mh]}\right)\right],$

$= \frac{1}{n_h - 1}\left[\frac{n_h}{N_h}\sum_{j=1}^{n_j} h_{h[i:mh]} z_{h[i:mh]j} - n_h\left(Cov\left(\overline{h}_{h[i:mh]} \overline{Z}_{h[i:mh]}\right) + \overline{H}_h \overline{Z}_h\right)\right],$

$= \frac{n_h}{n_h - 1}\left[\frac{1}{N_h}\sum_{j=1}^{n_j} h_{h[i:mh]} z_{h[i:mh]j} - \overline{H}\,\overline{Z} - \frac{s_{h[i:mh]} z_{[i:mh]}}{n_h}\right]$

$= \frac{n_h}{n_h - 1}\left(S_{h[i:mh]} z_{h[i:mh]} - \frac{s_{h[i:mh]} z_{[i:mh]}}{n_h}\right)$

$= S_{h[i:mh]} z_{h[i:mh]}$

An unbiased estimate of bias given in Eq (8) becomes

$$\hat{B}\left(\overline{y}_{(SRSS)}\right) = -\sum_{h=1}^{L} P_h\left[\frac{N_h - 1}{N_h}\left(\frac{n_h - 1}{n_h}\overline{Z}_{hsrg[i:mh]} + S_{hz[i:mh]}\right)\right]$$

Where, $S_{rg[i:m]} = \frac{1}{n-1}\sum_{j=1}^{n} g_{[i:m]j}\left(r_{[i:m]j} - \overline{r}\right)$ and $S_{hz[i:m]} = \frac{1}{n-1}\sum_{j=1}^{n} h_{[i:m]j}\left(z_{[i:m]j} - \overline{z}\right).$

$$\overline{y}_4^{(u)} = \sum_{h=1}^{L} P_h\left[\overline{r}_{h[rss]}\overline{g}_{h[rss]}\overline{Z}_h + \frac{N_h - 1}{N_h}\left(\frac{n_h - 1}{n_h}\overline{Z}_{hsrg[i:mh]} + s_{hz[i:mh]}\right)\right]$$

$= \sum_{h=1}^{L} P_h\left[\overline{r}_{h[rss]}\overline{g}_{h[rss]}\overline{Z}_h + \frac{n_h(N_h-1)}{N_h(n_h-1)}\left(\frac{n_h-1}{n_h}\left(\overline{h}_{h[rss]} - \overline{r}_{h[rss]}\overline{g}_{h[rss]}\right)\right)\overline{Z}_h + \left(\overline{y}_{h[rss]} - \overline{z}_{h[rss]}\overline{h}_{h[rss]}\right)\right]$

(9)

To find the variance of the unbiased estimators, we define the following error terms.

$\overline{y}_{h[rss]} = \overline{Y}_h(1 + \delta_{0h}), \overline{x}_{h[rss]} = \overline{X}_h(1 + \delta_{1h})\overline{r}_{h[rss]} = \overline{R}_h(1 + \delta_{2h}),$

$\overline{z}_{h[rss]} = \overline{Z}_h(1 + \delta_{3h}),\ \overline{h}_{h[rss]} = \overline{H}_h(1 + \delta_{4h}),\ \overline{g}_{h[rss]} = \overline{G}_h(1 + \delta_{5h}),$

Such that $\mathrm{E}\left(\delta_{ph}\right) = 0$, $(p = 0,1,2,3,4,5)$, *and*

$\mathrm{E}\left(\left(\delta_{0h}^2\right)\right) = \gamma_h C_{yh}^2 - W_{yh}^2$,$\mathrm{E}\left(\left(\delta_{1h}^2\right)\right) = \gamma_h C_{xh}^2 - W_{xh}^2$,$\mathrm{E}\left(\left(\delta_{2h}^2\right)\right) = \gamma_h C_{rh}^2 - W_{rh}^2$,

$\mathrm{E}\left(\left(\delta_{3h}^2\right)\right) = \gamma_h C_{zh}^2 - W_{zh}^2$,$\mathrm{E}\left(\left(\delta_{4h}^2\right)\right) = \gamma_h C_{hh}^2 - W_{hh}^2$,$\mathrm{E}\left(\left(\delta_{5h}^2\right)\right) = \gamma_h C_{gh}^2 - W_{gh}^2$,

$\mathrm{E}(\delta_{0h}\delta_{3h}) = \gamma_h C_{yhzh} - W_{yhzh}$, $\mathrm{E}(\delta_{0h}\delta_{5h}) = \gamma_h C_{yhgh} - W_{yhgh}$

$\mathrm{E}(\delta_{3h}\delta_{5h}) = \gamma_h C_{yhgh} - W_{yhgh}$

Where,

$W_{yh}^2 = \frac{1}{m_h^2 r \overline{Y}_h^2} \sum_{i=1}^{m_h} \tau_{yh[i:mh]}^2$, $W_{xh}^2 = \frac{1}{m_h^2 r \overline{X}_h^2} \sum_{i=1}^{m_h} \tau_{x[i:mh]}^2$, $W_{rh}^2 = \frac{1}{m_h^2 r \overline{R}_h^2} \sum_{i=1}^{m_h} \tau_{r[i:mh]}^2$

$W_{zh}^2 = \frac{1}{m_h^2 r \overline{Z}_h^2} \sum_{i=1}^{m_h} \tau_{z[i:mh]}^2$, $W_{hh}^2 = \frac{1}{m_h^2 r \overline{H}_h^2} \sum_{i=1}^{m_h} \tau_{hh[i:mh]}^2$, $W_{gh}^2 = \frac{1}{m_h^2 r \overline{G}_h^2} \sum_{i=1}^{m_h} \tau_{gh[i:mh]}^2$

$W_{yh\,zh} = \frac{1}{m_h^2 r \overline{Z}_h \overline{Y}_h} \sum_{i=1}^{m_h} \tau_{yh\,zh(i:mh)}$, $W_{yh\,gh} = \frac{1}{m_h^2 r \overline{G}_h \overline{Y}_h} \sum_{i=1}^{m_h} \tau_{yh\,gh(i:mh)}$,

$W_{zh\,gh} = \frac{1}{m_h^2 r \overline{Z}_h \overline{G}_h} \sum_{i=1}^{m_h} \tau_{zh\,gh(i:mh)}$, $W_{yh\,xh} = \frac{1}{m_h^2 r \overline{Y}_h \overline{X}_h} \sum_{i=1}^{m_h} \tau_{yh\,xh(i:mh),}$

And

$\tau_{yh[i:mh]} = \left(\mu_{yh[i:mh]} - \overline{Y}_h\right)$, $\tau_{xh[i:mh]} = \left(\mu_{xh[i:mh]} - \overline{X}_h\right)$, $\tau_{rh[i:mh]} = \left(\mu_{rh[i:mh]} - \overline{R}_h\right)$

$\tau_{zh[i:mh]} = \left(\mu_{zh[i:mh]} - \overline{Z}_h\right)$, $\tau_{hh[i:mh]} = \left(\mu_{hh[i:mh]} - \overline{H}_h\right)$, $\tau_{gh[i:mh]} = \left(\mu_{gh[i:mh]} - \overline{G}_h\right)$

$\tau_{yh\,zh[i:mh]} = \left(\mu_{yh[i:mh]} - \overline{Y}_h\right)\left(\mu_{zh[i:mh]} - \overline{Z}_h\right)$, $\tau_{yh\,gh[i:mh]} = \left(\mu_{yh[i:mh]} - \overline{Y}_h\right)\left(\mu_{zh[i:mh]} - \overline{G}_h\right)$

$\tau_{zh\,gh[i:mh]} = \left(\mu_{yh[i:mh]} - \overline{Z}_h\right)\left(\mu_{zh[i:mh]} - \overline{G}_h\right)$, $\tau_{yh\,xh[i:mh]} = \left(\mu_{yh[i:mh]} - \overline{Y}_h\right)\left(\mu_{zh[i:mh]} - \overline{X}_h\right)$

Here, $C_{yh\,zh} = \rho_{yh\,zh} C_{yh} C_{zh}, C_{yh\,xh} = \rho_{yh\,xh} C_{yh} C_{xh},$

$$C_{yh\,gh} = \rho_{yh\,gh} C_{yh} C_{gh} C_{zh\,gh} = \rho_{zh\,gh} C_{zh} C_{gh}$$

C_{yh} , C_{zh} , *and* C_{gh} are the coefficients of variation of y_h, zh and g_h, respectively. The values of $\mu_{yh[i:mh]}$, $\mu_{xh[i:mh]}$, $\mu_{rh[i:mh]}$, $\mu_{zh[i:mh]}$, $\mu_{hh[i:mh]}$,

$\mu_{gh[i:mh]}$, and $\mu_{zh[i:mh]}$ depend on order statistics from some specific distributions (see for example Arnold et al.(1992)).

In terms of $\delta's$, *we have*

$$\overline{y}_4^u = \sum_{h=1}^{L} P_h \overline{R}_h \overline{G}_h \overline{Z}_h (1 + \delta_{2h})(1 + \delta_{5h}) \frac{n_h(N_h - 1)}{N_h(n_h - 1)} \left[\frac{(n_h - 1)}{n_h} \overline{Z}_h \overline{H}_h (1 + \delta_{4h}) - \overline{Z}_h \overline{R}_h \overline{G}_h (1 + \delta_{2h})(1 + \delta_{5h}) + \overline{Y}_h (1 + \delta_{0h}) - \overline{Z}_h \overline{H}_h (1 + \delta_{3h})(1 + \delta_{4h}) \right].$$

Under the assumptions $\frac{(N_h-1)}{N_h} \cong 1$ and $\frac{(n_h-1)}{n_h} \cong 1$ we can write

$$\overline{y}_4^{(u)} - \overline{Y} \cong \sum_{h=1}^{L} P_h\left[\left(\overline{Y}_h\delta_{0h} - \overline{Z}_h\overline{H}_h\delta_{3h}\right)\right]$$

Taking square and then expectation, the variance of $\overline{y}_{(S_tRss)}$ is given by

$$\mathrm{V}\left(\overline{y}_4^{(u)}\right) \cong \sum_{h=1}^{L} P_h^2 \left[\overline{Y}_h^2\left(\gamma_h C_{yh}^2 - W_{yh}^2\right) + \overline{Z}_h^2\overline{H}_h^2\left(\gamma_h C_{zh}^2 - W_{zh}^2\right) - 2\,\overline{Y}_h\overline{Z}_h\overline{H}_h\left(\gamma C_{yh\,zh} - W_{yh\,zh}\right)\right]. \qquad (10)$$

Efficiency Comparison

We obtain the conditions under which the proposed unbiased estimator $\overline{y}_4^{(u)}$ is more efficient than the usual RSS mean estimator $\overline{y}_1^{(u)}$ and HR type unbiased estimator $\overline{y}_2^{(u)}$ and $\overline{y}_3^{(u)}$

(i) By Eq (2) and Eq (10),

$$V\left(\overline{y}_4^{(u)}\right) < V\left(\overline{y}_1^{(u)}\right)$$

$$\sum_{h=1}^{L} P_h^2 \left[\overline{Z}_h^2\overline{H}_h^2\left(\gamma_h C_{zh}^2 - W_{zh}^2\right) - 2\overline{Y}_h\overline{Z}_h\overline{H}_h\left(\gamma C_{yh\,zh} - \mathrm{W}_{yh\,zh}\right)\right] < 0.$$

(ii) By Eq (4) and Eq (10),

$$V\left(\overline{y}_4^{(u)}\right) < V\left(\overline{y}_2^{(u)}\right)$$

$$\sum_{h=1}^{L} P_h^2 \left[\overline{Z}_h^2\overline{H}_h^2\left(\gamma_h C_{zh}^2 - W_{zh}^2\right) - 2\overline{Y}_h\overline{Z}_h\overline{H}_h\left(\gamma C_{yh\,zh} - \mathrm{W}_{yh\,zh}\right)\right] <$$

$$\sum_{h=1}^{L} P_h^2 \left[\overline{X}_h^2\overline{R}_h^2\left(\gamma_h C_{xh(k)}^2 - W_{xh(i:mh)}^2\right) - 2\overline{R}_h\overline{X}_h\overline{Y}_h\left(\gamma C_{yh\,xh} - \mathrm{W}_{yh\,xh(i:mh)}\right)\right],$$

(iii) By Eq (6) and Eq (10),

$$V\left(\overline{y}_4^{(u)}\right) < V\left(\overline{y}_3^{(u)}\right)$$

$$\sum_{h=1}^{L} P_h^2 \left[\overline{Z}_h^2\overline{H}_h^2\left(\gamma_h C_{zh}^2 - W_{zh}^2\right) - 2\overline{Y}_h\overline{Z}_h\overline{H}_h\left(\gamma C_{yh\,zh} - \mathrm{W}_{yh\,zh}\right)\right] <$$

$$\sum_{h=1}^{L} P_h^2 \left[\left(\beta_h + \overline{R}_h\right)\overline{X}_h\left(\gamma_h C_{xh}^2 - W_{xh}^2\right) - 2\left(1 + \overline{R}_h\right)\overline{X}_h\overline{Y}_h\left(\gamma C_{yh\,zh} - \mathrm{W}_{yh\,zh}\right)\right]$$

A simulation Study in SRSS

To compare the performances of the proposed estimators, a simulation study is conducted. Ranking is performed on the basis of the auxiliary variable X. Tri-variate random observations (X_h, Y_h, Z_h), h = 1, 2, . . ., L are generated from a tri-variate normal population whose population parameters are $\left(\mu_{xh}, \mu_{yh}, \mu_{zh}, \delta_{xh}, \delta_{yh}, \delta_{zh}, \rho_{yxh}\rho_{yzh}, \rho_{xzh}\right)$. . Here, L = 3, P_h = (.30, .30, .40), m_h = (3, 4, 5), r = (3, 4, 5, 10, 15, 20), μ_{xh} = (2, 3, 4), μ_{yh} = (1, 2, 3), μ_{zh} = (3, 4, 5), σ_{yh} = (1, 1.5, 2), σ_{xh} = (1, 1.5, 2) σ_{zh}= (1, 1.5, 2). On the basis of 20, 000 repetitions, estimates of Variances, P RB and percentage RRMSE are computed under stratified ranked sampling scheme as described in Section 2. Estimators are also compared in terms of PREs. We use the following expressions to obtain the Variances, P RB, P RRMSE and PREs.

$$\mathrm{PRB}\left(\overline{y}_i^{(u)}\right) = \frac{1}{\overline{\overline{Y}}}\left[\frac{1}{20000}\sum_{i=1}^{20000}\left(\overline{y}_i^{(u)} - \overline{Y}\right)\right] X\,100, \quad i = 1,2,3,4$$

$$\mathrm{V}\left(\overline{y}_i^{(u)}\right) = \frac{1}{20000}\sum_{i=1}^{20000}\left(\overline{y}_i^{(u)} - \overline{Y}\right)^2, \quad i = 1,2,3,4$$

$$\mathrm{PRRMSE}\left(\overline{y}_i^{(u)}\right) = \frac{1}{\overline{\overline{Y}}}\left[\frac{1}{20000}\sum_{i=1}^{20000}\left(\overline{y}_i^{(u)} - \overline{Y}\right)^2\right]^{\frac{1}{2}} X\,100, \quad i = 1,2,3,4$$

$$\mathrm{PRE}\left(\overline{y}_i^{(u)}\right) = \frac{\mathrm{V}\left(\overline{y}_i^{(u)}\right)}{\mathrm{V}\left(\overline{y}_i^{(u)}\right)} \mathrm{X}\ 100,\ \mathrm{i} = 1, 2, 3, 4.$$

Table 1: Simulated variances and (PREs) of different estimators (assuming equal standard deviation for each stratum).

L = 3, P_h = (.30, .30, .40), m_h= (3, 4, 5), r = (3, 4, 5, 10, 15, 20).

Table 2: Simulated RRMSE and (PRB) of different estimators (assuming equal standard deviation for each stratum).

n_h	$\bar{y}_1^{(u)}$	$\bar{y}_2^{(u)}$	$\bar{y}_3^{(u)}$	$\bar{y}_4^{(u)}$
9,12,15	0.03633 (100)	0.03175 (114)	0.03409 (106)	0.02709 (134)
12,16,20	0.02647 (100)	0.02228 (118)	0.02385 (110)	0.01863 (142)
15,20,25	0.01965 (100)	0.01653 (119)	0.01776 (111)	0.01424 (138)
30,40,50	0.01051 (100)	0.00925 (113)	0.01021 (104)	0.00810 (130)
45,60,75	0.00672 (100)	0.00590 (113)	0.00621 (108)	0.00517 (129)
60,80,100	0.00508 (100)	0.00458 (111)	0.00486 (104)	0.00386 (131)

L = 3, P_h = (.30, .30, .40), m_h= (3, 4, 5), r = (3, 4, 5, 10, 15, 20).

n_h	$\bar{y}_1^{(u)}$	$\bar{y}_2^{(u)}$	$\bar{y}_3^{(u)}$	$\bar{y}_4^{(u)}$
9,12,15	8.89 (0.11)	8.31 (0.12)	8.43 (0.14)	7.80 (0.08)
12,16,20	7.59 (-0.06)	6.97 (-0.22)	7.04 (0.18)	6.35 (0.03)
15,20,25	6.55 (0.42)	6.01 (0.36)	6.18 (0.38)	5.70 (0.21)
30,40,50	4.78 (-0.04)	4.49 (-0.07)	4.52 (-0.08)	4.20 (-0.03)
45,60,75	3.82 (-0.04)	3.58 (-0.13)	3.60 (0.09)	3.41 (-0.02)
60,80,100	3.33 (-0.02)	3.17 (-0.03)	3.22 (-0.03)	2.91 (-0.01)

Table 3: Simulated variances and (P REs) of different estimators (assuming unequal standard deviation for each stratum).

L = 3, P_h = (.30, .30, .40), m_h= (3, 4, 5), r = (3, 4, 5, 10, 15, 20).

n_h	$\bar{y}_1^{(u)}$	$\bar{y}_2^{(u)}$	$\bar{y}_3^{(u)}$	$\bar{y}_4^{(u)}$
9,12,15	0.1778 (100)	0.01354 (131)	0.01592 (111)	0.01044 (170)
12,16,20	0.01187 (100)	0.00944 (125)	0.01094 (108)	0.00677 (175)
15,20,25	0.00995 (100)	0.00781 (127)	0.00921 (109)	0.00508 (195)
30,40,50	0.00530 (100)	0.00409 (129)	0.00447 (118)	0.00285 (185)
45,60,75	0.00354 (100)	0.00253 (139)	0.00315 (112)	0.00208 (170)
60,80,100	0.00241 (100)	0.00193 (125)	0.00211 (114)	0.00131 (184)

Table 4: Simulated RRMSE and (P RB) of different estimators (assuming unequal standard deviation for each stratum).

$L = 3$, $P_h = (.30, .30, .40)$, $m_h = (3, 4, 5)$, $r = (3, 4, 5, 10, 15, 20)$.

n_h	$\bar{y}_1^{(u)}$	$\bar{y}_2^{(u)}$	$\bar{y}_3^{(u)}$	$\bar{y}_4^{(u)}$
9,12,15	6.02 (-0.15)	5.26 (0.18)	5.65 (0.20)	4.61 (0.02)
12,16,20	4.92 (0.23)	4.38 (0.11)	4.41 (0.16)	3.70 (0.07)
15,20,25	4.55 (0.11)	3.92 (0.23)	3.90 (0.21)	3.20 (0.10)
30,40,50	3.29 (-0.18)	2.90 (-0.23)	3.01 (-0.29)	2.42 (-0.16)
45,60,75	2.68 (0.02)	2.27 (-0.13)	2.33 (-0.18)	2.06 (-0.03)
60,80,100	2.22 (-0.10)	2.00 (-0.08)	2.05 (-0.11)	1.62 (-0.01)

CONCLUSION

The simulated variances and PRE values are calculated for different sample size assuming equal standard deviation for each stratum and are shown in Table 1. The results of Table 2 showed the simulated values of RRMSE and PRB for equal standard deviation having different stratum size. Similarly, Tables 3 and 4 showed the simulated values of variances, PRE, RRMSE and PRB for different sample size assuming unequal standard deviation among strata. The proposed estimator has minimum variance and has greater PRE values as compare to competitor estimator. The new suggested estimator has minimum values of RRMSE and PRB. It is observed that the newly suggested Hartly-Ross type estimator is more efficient as compared to all the considered competitor estimators under SRSS design.

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