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Performance of Various Entropy Measures: Applications to Pareto and Truncated Pareto Distributions

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ABSTRACT

This study considers various entropy measures for Pareto distribution and Truncated Pareto distribution, and also calculates the loss of entropy when underlying distribution is truncated Pareto distribution instead of Pareto distribution. The mathematical expression of entropy measures was derived for Pareto and Truncated Pareto distribution. Then the mathematical expression of relative loss was derived to check the performance of various entropy measures. For Comparison purpose the study considers a real data set available in the literature and find out the maximum likelihood estimate for the parameters of Pareto distribution. The results of the relative loss of entropy measures showed that the natural phenomenon holds in Shannon, Awad, Renyi and Harvrd & Charvat entropy measures for Pareto distribution, while in Awad et al. and Arimoto's it does not hold. Amongst the four entropy measures the Shannon entropy measure is considered best because it gives us the minimum loss of information if one considers Truncated Pareto distribution instead of Pareto distribution.

Keywords: *Entropy, Pareto distribution, Relative loss, Shannon, Probability*

INTRODUCTION

Probability is a statistical measure used for quantification of uncertainty. This terminology is of keen interest while studying uncertainty in a phenomenon and has a wide application in different fields of science etc. Probability helps to study the experiment which is random in nature. The term probability is a backbone of statistics and has a key role in statistical inference. Probability distribution helps to model the real-life phenomenon involving uncertainty and to study its statistical characteristics (Schiller et.al. 2012). A function that enables us to compute the probabilities of a random variable is called probability distribution function. There are many processes in science and other

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fields, which can be probabilistically, described using probability distribution (Jaynes, 2003).

Probability distribution has its own purpose and represents the various data generated process. There are many probability distributions like binomial, exponential, geometric, exponential, normal and some others depending upon the discrete and continuous nature of the random variable, which has wide applications in different fields. Sharma & Singh (2012) proposed generalized extreme value distribution for the analysis of rainfall data. The probability distributions also help in modelling lifetime phenomena, like, Exponential, Gamma, Weibull, and Pareto. The statistical characteristics of Pareto distribution Moment Generating Function, characteristics function and some other properties have been discussed in the literature, whereas an important statistical characteristic, i.e. (entropy) of the distribution is considered in this study.

Pareto Distribution

The idea of Pareto distribution was given by Italian Economist, Vilfredo Pareto in 1906. It is a power law probability distribution that has an application in observable phenomena of various fields, such as, social, scientific research area. This distribution was applied in economical phenomena, that is, wealth in society. Sometimes this distribution in the field of economics is also known as "80-20 rule" or "Matthew principle". This means 80% of the wealth in a society is held by 20% of the entire population. Pareto distribution is a continuous heavy tailed distribution in nature. For nonnegative data, it is simple to model with a power law probability tail (Aban, et.al. 2006).

Thermodynamic

The branch of physics which deals with heat and temperature, and their relation to energy, work and properties of matter. There are four laws of a thermodynamics states. While the second law of thermodynamic state deals with the concept of entropy (Belkin, 2005).

Entropy

Term Entropy originated from Thermodynamic system in 19th century and applied in various fields like biology, economics, statistics, information theory, etc. Czogala & Leski (1998) used the entropy measure in processing the Electrocardiograph (ECG) signals. It measures the disorder of the thermodynamic system, i.e. it measures the number of all possible ways in which a thermodynamic system can be arranged (Bailey, 2009). In 1948 Shannon liken the term entropy with information theory with the goal of finding a measure for the degree of novelty of a message during communication in a noisy channel. Generally in statistics, especially in probability, it is used to measures the amount of uncertainty in a data set modelled by using an appropriate probability distribution (Dey, 2016). There are two phenomena, first the gain in entropy or uncertainty means loss of information, whereas reduction in entropy or uncertainty means a gain in information (without ignoring or losing of data) (Garrido, 2011). This information is related to the occurrence of an event, the second one is the natural one, i.e, a gain in entropy or uncertainty means a gain in information, whereas reduction in the entropy or uncertainty means losing of information. Various entropy measures have been derived by Shannon (1948), Renyi (1961), Havrda & Charvat (1967), Awad (1987), Awad, et. al. (1987), Arimoto's (1971).

A huge literature is available on the Entropy measures and principles along with its application in different fields.

Shannon (1948) proposed an entropy measure for the first time and the goal was to measure the degree of novelty of a message during communication in a noisy channel. Later on various entropy measures were derived by Renyi (1961), Havrda & Charvat (1967), Awad (1987), Awad et. al., (1987), Arimoto's (1971).

Zhou et. al., (2013) used the concept of entropy in finance, especially for Portfolio selection and asset pricing. The Study considered entropy as a measure of risk in portfolio and used principles of entropy in asset pricing, to tackle out the problem of incomplete market information.

Zanin et. al., (2012) used the idea of calculating entropy based on permutation patterns to complex systems. The Study derived the theoretical form of permutation entropy from existing measures in a time series and used its application in the analysis of economic markets and biomedical systems. The study showed the application of this method in time series analysis, such as classifying different dynamics, identifying break points, predicting future events etc. In addition, this method is helpful to tackle with simple scalar time series, which can be extended to multi-variates and multi-scale systems.

Award et. al., (1987) proposed a mathematical expression of six entropy measures to calculate the relative loss of truncated Exponential on [0, t] in place of Exponential on $[0, \infty]$. The performance of various entropy measures had been checked, i.e, empirically the relative loss was calculated. The study showed that the truncation time t increases the relative loss of A-entropy measures decreases, while for H-entropy in some cases the natural phenomenon did not hold. Thus, the A-entropy measures have advantages over the H-entropy measures.

Dey et. al., (2016) derived a mathematical expression of seven entropy measures to calculate the relative loss for life time distribution using truncated Rayleigh on [0, t] instead of Rayleigh on $[0, \infty]$. Numerical results were used to study the performance of various entropy measures in terms of relative loss. The H-entropy measures were better that of Awad's entropy.

Basit et. al., (2017) suggested mathematical expression of various entropy measures to calculate the relative loss when the distribution is truncated size biased exponential rather than size biased exponential distribution. To compare the performance of various entropy measures, the relative loss calculated numerically. The relative loss of Awad's entropy measures were increases as the truncation time t increases, which is not a natural phenomenon, while for other entropy measures the natural phenomena hold as truncation time t increases the relative loss decreases. The study showed that the Awad's entropy measures were not better that of others.

Mahdy & Eltelbany (2017) derived Differential and Beta Entropy for Nakagami-U distribution and its selected versions. The Relative loss of Nakagami-U distribution and it's all selected version were calculated. The study also compared the numerical performance of relative loss and entropy measures for the selected version of Nakagami-U.

Chen et. al., (2011) suggested a new minimum error entropy type adaption criterion, called the order alpha criterion survival information potential base on survival function, while minimum error entropy is the best criterion for adaption system training.

Frery et. al., (2012) derived a mathematical expression for three entropy measures, and these are, Shannon, Reyni and Tsallis when underlying distribution was scaled complex Wishart distribution. The study derived variances and hypothesis test for entropies. To check the performance of suggested hypothesis test, the Monte Carlo experiments were employed. The actual data was considered and check the performance. The results showed that the Shannon entropy can be carefully used for sharp areas in PolSAR imagery.

Moradi et. al., (1998) used the concept of entropy in the English language after Shannon. The study used entropy as an instrument to measure the amount of information created on the average for each letter of a text of an English language. The study considered 100 passages 64 characters in length were selected from each of 2 books, while Shannon considers 100 passages only from one book with 15 characters. They ask the subjects to guess the next letter of a character and note all these guesses in binary codes to make the analysis. For analysis the Shannon proposed instrument was used to calculate the upper bound of entropy.

Ebrahimi et. al., (1990) considered the two most important concepts of statistics, i.e., Entropy and variance. Both are the measure of uncertainty and dispersion. The Study observed the role of entropy and variance in ordering distributions and random scenario respectively. The concept of dispersion order was used to make the relationship between variance and entropy. The general considerations and specific results showed that entropy based on more information rather than its variance.

Lee et. al., (2007) derived a new four parameter distribution 'Beta-Weibull Distribution'. The study discussed its application and proposed various properties for 'Beta-Weibull Distribution' including entropy measures. Mathematical expression for two entropy measures Shannon and Reyni were derived.

Abbasnejad (2011) proposed failure entropy is the best measure in case of the survived unit up to age t, while Shannon and Reyni entropy were no more a best measure of uncertainty. Also, study its properties and dynamic version.

Antoniou et. al., (2002) proposed the method of density functions to present economic systems with a little number of components and used entropy as an indicator of competence of the resources sharing. The proposed method is not restricted by the number of components of the economic system and useful to a broad class of economic problems.

Significance of Study

The consideration of this research work is to derive different entropy measures for Pareto distribution and truncated Pareto distribution. The expression of the relative loss of entropy will be derived and used to compare the performance of various entropy measures numerically. As a result of the truncation of distribution, the less affected entropy measure will be determined by calculating the relative loss of entropy.

METHODOLOGY

This article contains the research methodology used to achieve the mentioned research objectives. It includes a brief introduction about Pareto, truncated Pareto distribution, the Entropy and relative loss. Also, this article includes the proposed expression of Entropy and relative loss for Pareto and truncated Pareto distribution respectively. It includes the description of real data and Statistical software used for obtaining the maximum likelihood of the Pareto and truncated Pareto distribution respectively.

Pareto Distribution

A continuous random variable X has a Pareto distribution if its probability can be obtained by following function (PDF).

$$f(x;\theta,\lambda)=\frac{\lambda\theta^{\lambda}}{x^{\lambda+1}} \quad ,\theta\le x\le\infty \qquad \theta,\lambda>0$$

The cumulative distribution function (CDF) of *X* is given by

$$F(x;\theta,\lambda)=1-\frac{\theta^{\lambda}}{x^{\lambda}}.$$

Truncated Pareto Distribution

A continuous random variable Y is said to follow the truncated Pareto distribution if its probabilities can be obtained from the following defined PDF

$$f(y;t;\theta,\lambda)=\frac{f(y;\theta,\lambda)}{F(t;\theta,\lambda)}=\frac{\lambda\theta^{\lambda}/y^{\lambda+1}}{1-\theta^{\lambda}/t^{\lambda}} \quad ,\theta\le y\le t \quad \theta,\lambda>0$$

The CDF of the truncated Pareto distribution is given by

$$F(y;t;\theta,\lambda)=\frac{F(y;\theta,\lambda)}{F(t;\theta,\lambda)}=\frac{1-\theta^{\lambda}/y^{\lambda}}{1-\theta^{\lambda}/t^{\lambda}}.$$

Entropy Measures

Various entropy measures have been derived by Shannon (1948), Renyi (1961), Havrda & Charvat (1967), Awad (1987), Awad et. al.,(1987), Arimoto's (1971).

Entropy Measure by Shannon

Let random variable X; with PDF $f(x)$, the entropy measure denoted by $H(X)$ and defined by Shannon (1948)

$$H(X)=-\int_{R_x} f(x)\ln f(x)dx$$

Where,

$$R_x=(x:f(x)\neq 0).$$

Entropy Measure by Renyi

Let random variable X; with PDF $f(x)$, the entropy measure denoted by $H_\alpha(X)$ and defined by Renyi entropy (1961)

$$H_\alpha(X)=\frac{1}{1-\alpha}\ln\int_{R_x} f^{\alpha}(x)dx; \quad \alpha>0,\alpha\neq 1$$

Entropy Measure byHavrda and Charvat

Let random variable X; with PDF $f(x)$, the entropy measure denoted by $H^{\alpha}(X)$ and defined by Havrda and Charvat entropy (1967).

$$H^{\alpha}(X) = \frac{1}{2^{1-\alpha} - 1} \left(\int_{R_x} f^{\alpha}(x) dx - 1 \right); \quad \alpha > 0$$

Entropy Measure by Awad

Let random variable X; with PDF $f(x)$, the entropy measure denoted by $A(X)$ and defined by Awad entropy (1987)

$$A(X) = -\int_{R_x} f(x) \ln \frac{f(x)}{\delta} dx$$

Where,

$$\delta = \sup_{x \in R_x} f(x)$$

Entropy Measure by Awad et. al.

Let random variable X; with PDF $f(x)$, the entropy measure denoted by $_{\alpha}A(X)$ and defined by Awad et al entropy [16]

$$_{\alpha}A(X) = \frac{1}{1-\alpha} \ln \int_{R_x} \left(\frac{f(x)}{\delta} \right)^{\alpha-1} f(x) \, dx$$

Entropy Measure by Arimoto's

Let random variable X; with PDF $f(x)$, the entropy measure denoted by $A_{\alpha}(X)$ and defined by Arimoto's entropy (1971)

$$A_{\alpha}(X) = \frac{1}{2^{\alpha-1} - 1} \left[\left\{ \int_{R_x} \left(f(x) \right)^{\frac{1}{\alpha}} dx \right\}^{\alpha} - 1 \right]; \quad \alpha > 0, \alpha \neq 1$$

To check that which one of the entropy measures are perform well using a specific probability distribution. We have to measure the relative loss for each entropy measures.

Relative Loss

Let I(X) and I(Y) are the two entropies of the Pareto distribution and truncated Pareto distribution respectively. Then the relative loss of entropy, taking Y instead of X is defined as

$$S_I(t) = \frac{I(X) - I(Y)}{I(X)}$$

For each entropy measure the expression of the relative loss will be obtained using above expression.

Proposed Entropy Measures and Relative Loss for Pareto and Truncated Pareto Distribution

The following are the proposed expressions of entropy measures for pareto and truncated pareto distribution respectively. The detail derivations are given in the Appendix 1 and Appendix 2 respectively.

Shannon Entropy Measures

$$H(X) = -\int_{\theta}^{\infty} f(x;\theta,\lambda) \ln f(x;\theta,\lambda)dx$$

$$H(x) = 1 + \frac{1}{\lambda} + \ln\theta - \ln\lambda$$

Similarly, for truncated Pareto distribution

$$H(\text{Y}) = -\int_{\theta}^{t} f(y;t;\theta,\lambda) \ln f(y;t;\theta,\lambda)dy$$

$$= -\int_{\theta}^{t} \frac{f(y;\theta,\lambda)}{F(t;\theta,\lambda)} \ln \frac{f(y;\theta,\lambda)}{F(t;\theta,\lambda)} dy$$

$$H(Y)=\frac{t^{\lambda}}{t^{\lambda}-\theta^{\lambda}}\left[\begin{array}{l}\left\{\left(1-\theta^{\lambda}t^{-\lambda}\right)\ln\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)\right\}-\left\{\ln\lambda\theta^{\lambda}-\theta^{\lambda}t^{-\lambda}\ln\lambda\theta^{\lambda}\right\}-\\ \left\{\frac{\left(\lambda\theta^{\lambda}+\theta^{\lambda}\right)}{\lambda}\left\{t^{-\lambda}+\lambda t^{-\lambda}\ln t-\theta^{-\lambda}-\lambda\theta^{-\lambda}\ln\theta\right\}\right\}\end{array}\right]$$

Relative Loss for Shannon Measure

The Relative loss of the Shannon Entropy when using Y instead of X

$$S_H(t)=\frac{\mathrm{H}(X)-H(Y)}{\mathrm{H}(X)}$$

OR

$$S_H(t)=1-\frac{H(Y)}{\mathrm{H}(X)}$$

$$S_H(t)=1-\left[\frac{\frac{t^{\lambda}}{t^{\lambda}-\theta^{\lambda}}\left[\begin{array}{l}\left\{\left(1-\theta^{\lambda}t^{-\lambda}\right)\ln\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)\right\}-\left\{\ln\lambda\theta^{\lambda}-\theta^{\lambda}t^{-\lambda}\ln\lambda\theta^{\lambda}\right\}-\\ \left\{\frac{\left(\lambda\theta^{\lambda}+\theta^{\lambda}\right)}{\lambda}\left\{t^{-\lambda}+\lambda t^{-\lambda}\ln t-\theta^{-\lambda}-\lambda\theta^{-\lambda}\ln\theta\right\}\right\}\end{array}\right]}{1+\frac{1}{\lambda}+\ln\theta-\ln\lambda}\right]$$

Reyni Entropy Measures

The Reyni Entropy of X and Y is given by

$$H_{\alpha}(X)=\frac{1}{1-\alpha}\ln\int_{\theta}^{\infty}f^{\alpha}(x;\theta,\lambda)dx;\quad \alpha>0,\alpha\neq 1$$

$$H_\alpha(X) = \frac{1}{1-\alpha}\left[\alpha \ln \lambda + (1-\alpha)\ln\theta - \ln\left\{\alpha\left(\lambda+1\right)+1\right\}\right]$$

Similarly, for Truncated Pareto distribution

$$H_\alpha(Y) = \frac{1}{1-\alpha}\ln\int_\theta^t f^\alpha(y;t;\theta,\lambda)dy; \quad \alpha > 0, \alpha \neq 1$$

$$H_\alpha(Y) = \frac{1}{1-\alpha}\left[\alpha \ln \lambda + \ln\left(t^{\lambda\alpha}\theta^{1-\alpha} - \theta^{\lambda\alpha}t^{1-\alpha}\right) - \alpha\ln\left(t^\lambda - \theta^\lambda\right) - \ln\left\{\alpha\left(\lambda+1\right)+1\right\}\right]$$

Relative Loss for Renyi Measure

The Relative loss of Reyni Entropy when using Y instead of X

$$S_{H_\alpha}(t) = \frac{\mathrm{H}_\alpha(X) - H_\alpha(Y)}{\mathrm{H}_\alpha(X)}$$

$$S_{H_\alpha}(t) = \frac{(1-\alpha)\ln\theta + \alpha\ln\left(t^\lambda - \theta^\lambda\right) - \ln\left\{t^{\lambda\alpha}\theta^{1-\alpha} - \theta^{\lambda\alpha}t^{1-\alpha}\right\}}{\alpha\ln\lambda + (1-\alpha)\ln\theta - \ln\left\{\alpha\left(\lambda+1\right)+1\right\}}$$

Harvrda and Charvat Entropy Measures

Harvrda and Charvat proposed Entropy expression for the Pareto distribution and is defined by

$$H^\alpha(X) = \frac{1}{2^{1-\alpha}-1}\left[\int_\theta^\infty f^\alpha(x;\theta,\lambda)dx - 1\right]; \quad \alpha > 0$$

$$H^\alpha(X) = \frac{1}{2^{1-\alpha}-1}\left[\frac{\lambda^\alpha\theta^{1-\alpha} - \left\{\alpha\left(\lambda+1\right)+1\right\}}{\alpha\left(\lambda+1\right)+1}\right]$$

Also, Harvrda and Charvat proposed Entropy expression for the truncated Pareto distribution is

$$H^{\alpha}(Y) = \frac{1}{2^{1-\alpha}-1}\left(\int_{\theta}^{t} f^{\alpha}(y;t;\theta,\lambda)dy - 1\right); \quad \alpha > 0$$

$$H^{\alpha}(Y) = \frac{1}{2^{1-\alpha}-1}\left[\frac{\lambda^{\alpha}t^{\lambda\alpha}\theta^{1-\alpha} - \lambda^{\alpha}\theta^{\lambda\alpha}t^{1-\alpha} - \left(t^{\lambda}-\theta^{\lambda}\right)^{\alpha}\left\{\alpha\left(\lambda+1\right)+1\right\}}{\left(t^{\lambda}-\theta^{\lambda}\right)^{\alpha}\left\{\alpha\left(\lambda+1\right)+1\right\}}\right]$$

Relative Loss for Harvrda and Charvat Measure

The Relative loss of Harvrda and Charvat Entropy when using Y instead of X

$$S_{H^{\alpha}}(t) = \frac{\mathrm{H}^{\alpha}(X) - H^{\alpha}(Y)}{\mathrm{H}^{\alpha}(X)}$$

$$S_{H^{\alpha}}(t) = 1 - \frac{H^{\alpha}(Y)}{\mathrm{H}^{\alpha}(X)}$$

$$S_{H^{\alpha}}(t) = 1 - \left[\frac{\lambda^{\alpha}t^{\lambda\alpha}\theta^{1-\alpha} - \lambda^{\alpha}\theta^{\lambda\alpha}t^{1-\alpha} - \left[\left(t^{\lambda}-\theta^{\lambda}\right)^{\alpha}\left\{\alpha\left(\lambda+1\right)+1\right\}\right]}{\left[\left(t^{\lambda}-\theta^{\lambda}\right)^{\alpha}\left\{\left(\lambda^{\alpha}\theta^{1-\alpha}\right) - \left\{\alpha\left(\lambda+1\right)+1\right\}\right\}\right]}\right]$$

Awad Entropy Measures

Awad proposed Entropy expression for Pareto distribution is

$$A(X) = -\int_{\theta}^{\infty} f(x;\theta,\lambda)\ln\frac{f(x;\theta,\lambda)}{\delta}dx$$

$$A(X) = 1 + \frac{1}{\lambda} + \ln\theta + \ln\delta - \ln\lambda$$

Also, Awad proposed Entropy expression for truncated Pareto distribution is

$$A(Y) = -\int_{\theta}^{t} f(y;t;\theta,\lambda)\ln\frac{f(y;t;\theta,\lambda)}{\delta}dy$$

$$= -\int_{\theta}^{t} \frac{f(y;\theta,\lambda)}{F(t;\theta,\lambda)} \ln \frac{f(y;\theta,\lambda)}{\delta F(t;\theta,\lambda)} dy$$

$$A(Y) = \frac{t^{\lambda}}{t^{\lambda}-\theta^{\lambda}} \left[\begin{array}{l} \left\{ \left(1-\theta^{\lambda}t^{-\lambda}\right) \ln \delta \left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}} \right) \right\} - \left\{ \ln \lambda\theta^{\lambda} - \theta^{\lambda}t^{-\lambda} \ln \lambda\theta^{\lambda} \right\} - \\ \left\{ \frac{\left(\lambda\theta^{\lambda}+\theta^{\lambda}\right)}{\lambda} \left\{ t^{-\lambda} + \lambda t^{-\lambda} \ln t - \theta^{-\lambda} - \lambda\theta^{-\lambda} \ln \theta \right\} \right\} \end{array} \right]$$

Relative Loss for Awad Measure

The Proposed Relative loss of Awad Entropy when using Y instead of X

$$S_A(t) = \frac{A(X) - A(Y)}{A(X)}$$

$$S_A(t) = 1 - \frac{A(Y)}{A(X)}$$

$$S_A(t) = 1 - \left[\frac{ \frac{t^{\lambda}}{t^{\lambda}-\theta^{\lambda}} \left[\begin{array}{l} \left\{ \left(1-\theta^{\lambda}t^{-\lambda}\right) \ln \delta \left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}} \right) \right\} - \left\{ \ln \lambda\theta^{\lambda} - \theta^{\lambda}t^{-\lambda} \ln \lambda\theta^{\lambda} \right\} - \\ \left\{ \frac{\left(\lambda\theta^{\lambda}+\theta^{\lambda}\right)}{\lambda} \left\{ t^{-\lambda} + \lambda t^{-\lambda} \ln t - \theta^{-\lambda} - \lambda\theta^{-\lambda} \ln \theta \right\} \right\} \end{array} \right] }{ 1 + \frac{1}{\lambda} + \ln \theta + \ln \delta - \ln \lambda } \right]$$

Awad et al., Entropy Measures

Awad et al proposed Entropy expression for Pareto distribution is

$${}_{\alpha}A(X) = \frac{1}{1-\alpha} \ln \int_{\theta}^{\infty} \left(\frac{f(x;\theta,\lambda)}{\delta} \right)^{\alpha-1} f(x;\theta,\lambda) dx$$

$$_{\alpha}A(X)=\frac{1}{1-\alpha}\left[\alpha\ln\lambda+(1-\alpha)\ln\theta-(\alpha-1)\ln\delta-\ln\left\{\alpha(\lambda+1)+1\right\}\right]$$

Also, Awad et al. proposed Entropy expression for truncated Pareto distribution is

$$_{\alpha}A(Y)=\frac{1}{1-\alpha}\ln\int_{\theta}^{t}\left(\frac{f(y;t;\theta,\lambda)}{\delta}\right)^{\alpha-1}f(y;t;\theta,\lambda)dy$$

$$_{\alpha}A(Y)=\frac{1}{1-\alpha}\begin{bmatrix}\ln\left(\lambda^{\alpha}\theta^{1-\alpha}-\lambda^{\alpha}\theta^{\lambda\alpha}t^{-\alpha(\lambda+1)+1}\right)-(\alpha-1)\ln\delta-\alpha\ln\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)- \\ \ln\left\{\alpha(\lambda+1)+1\right\}\end{bmatrix}$$

Relative Loss for Awad et al Measure

The Proposed Relative loss of Awad et al Entropy when using Y instead of X

$$S_{_{\alpha}A}(t)=\frac{_{\alpha}A(X)-{}_{\alpha}A(Y)}{_{\alpha}A(X)}$$

$$S_{_{\alpha}A}(t)=\frac{\left[\alpha\ln\lambda+(1-\alpha)\ln\theta+\alpha\ln\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)-\ln\left(\lambda^{\alpha}\theta^{1-\alpha}-\lambda^{\alpha}\theta^{\lambda\alpha}t^{-\alpha(\lambda+1)+1}\right)\right]}{\left[\alpha\ln\lambda+(1-\alpha)\ln\theta-(\alpha-1)\ln\delta-\ln\left\{\alpha(\lambda+1)+1\right\}\right]}$$

Arimoto's Entropy Measures

Arimoto's proposed Entropy expression for Pareto distribution is

$$A_{\alpha}(X)=\frac{1}{2^{\alpha-1}-1}\left[\left\{\int_{\theta}^{\infty}f^{\frac{1}{\alpha}}(x;\theta,\lambda)\,\mathrm{dx}\right\}^{\alpha}-1\right];\alpha>0,\alpha\neq1$$

$$A_{\alpha}(X)=\frac{1}{2^{\alpha-1}-1}\left[\frac{\lambda\theta^{\alpha-1}\alpha^{\alpha}-(\alpha+\lambda+1)^{\alpha}}{(\alpha+\lambda+1)^{\alpha}}\right]$$

Also, Arimoto's proposed Entropy expression for truncated Pareto distribution is

$$A_{\alpha}(Y)=\frac{1}{2^{\alpha-1}-1}\left[\left\{\int_{\theta}^{t} f^{\frac{1}{\alpha}}(\mathrm{y};\mathrm{t};\theta,\lambda)\,\mathrm{dy}\right\}^{\alpha}-1\right];\alpha>0,\alpha\neq 1$$

$$A_{\alpha}(Y)=\frac{1}{2^{\alpha-1}-1}\left[\frac{\alpha^{\alpha}\lambda\theta^{\alpha-1}-\alpha^{\alpha}\lambda\theta^{\lambda}t^{\alpha-\lambda-1}-\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)(\alpha+\lambda+1)^{\alpha}}{\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)(\alpha+\lambda+1)^{\alpha}}\right]$$

Relative Loss for Arimoto's Measure

The proposed expression for Relative loss of Arimoto's Entropy is

$$S_{A_{\alpha}}(t)=\frac{A_{\alpha}(X)-A_{\alpha}(Y)}{A_{\alpha}(X)}$$

$$S_{A_{\alpha}}(t)=1-\frac{A_{\alpha}(Y)}{A_{\alpha}(X)}$$

$$S_{A_{\alpha}}(t)=1-\left[\frac{\alpha^{\alpha}\lambda\theta^{\alpha-1}-\alpha^{\alpha}\lambda\theta^{\lambda}t^{\alpha-\lambda-1}-\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)(\alpha+\lambda+1)^{\alpha}}{\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)\left\{\lambda\theta^{\alpha-1}\alpha^{\alpha}-(\alpha+\lambda+1)^{\alpha}\right\}}\right]$$

Description of Data

To estimate the maximum likelihood, estimate and to check the performance of proposed entropy measure by the relative loss empirically, the real data set on rainfall of India (Jaynes, 2003) is considered in the study.

EMPIRICAL ANALYSIS

This article contains the estimation of the maximum likelihood estimation of the parameters for the Pareto distribution and truncated Pareto distribution, these parameters of Pareto and truncated Pareto distribution are done by using real world data named as "rainfall data" (Jaynes, 2003). The estimated parameters for the Pareto and truncated Pareto parameters were then used to estimate the entropy measures empirically for various level of loss of information. Also, in this article relative loss estimated by using proposed entropy measures for Pareto and truncated Pareto distribution. All the numerical estimations were done in Mathematica software. Following are the detailed description of entropy measures in terms of the relative loss along with the maximum likelihood estimates.

Empirical analysis of proposed Entropy measure by relative loss

Table 1: Relative loss of Shannon and Awad Entropy

t	**θ=0.01 λ=0.03**	
	$S_H(t)$	**$S_A(t)$**
0.5	1.03984	1.2071
1	1.02495	1.18982
1.5	1.01661	1.18013
2	1.01083	1.17342
2.5	1.00641	1.1683
3	1.00285	1.16416
3.5	0.999863	1.16069
4	0.997296	1.15772
4.5	0.995047	1.1551
5	0.993047	1.15278

The empirical results of the relative loss of the Shannon and Awad entropy are given in table 1. The results showed that the natural phenomenon holds for both the entropy measures, i.e., as the t increases the relative loss get decrease. While the results showed that, in some cases, the relative loss for the Shannon entropy is less than one, which is not the case in Awad entropy. So here, the Shannon entropy considers the best entropy measure for the Pareto distribution.

Table 2: Relative loss of Renyi and Harvrd & Charvat Entropy

t	**θ=0.01 λ=0.03 α=2**	
	$S_{H\alpha}(t)$	**$S_H^{\alpha}(t)$**
0.5	1.24357	2.40183
1	1.15919	1.77585
1.5	1.11545	1.51771
2	1.08654	1.36768
2.5	1.06521	1.26637
3	1.04844	1.1919
3.5	1.03469	1.13405
4	1.02308	1.08736
4.5	1.01307	1.04859
5	1.00429	1.01569

The empirical results of the relative loss of Renyi and Harvrd & Charvat entropy are given in table 2. The results showed that the natural phenomenon holds for both the entropy measures, i.e., as the t increases the relative loss get decrease. While the results showed that, in all cases, the relative loss for Renyi and Harvrd & Charvat entropy are more than one. These entropy measures do not provide a better result to the Pareto distribution as compared to the Shannon entropy.

Table 3: Relative loss of Renyi and H&C Entropy

t	**θ=0.01 λ=0.03 α=3**	
	$S_{H\alpha}(t)$	$S_H^{\alpha}(t)$
0.5	2.42893	51.9668
1	2.26019	32.8247
1.5	2.17361	25.9275
2	2.11664	22.1978
2.5	2.0747	19.7989
3	2.04177	18.0981
3.5	2.01482	16.8146
4	1.99208	15.8029
4.5	1.97249	14.9796
5	1.9553	14.2929

The empirical results of the relative loss of Renyi and Harvrd & Charvat entropy are given in table 3. The results showed that the natural phenomenon holds for both the entropy measures, i.e., as the t increases the relative loss get decrease. While the results showed that, in all cases, the relative loss for Renyi and Harvrd & Charvat entropy are more than one using a constant value for alpha=3. These two entropy measures do not lead to a flexible result as compared to the Shannon entropy.

Table 4: Relative loss of Renyi and H&C Entropy

t	**θ=0.01 λ=0.03 α=4**	
	$S_{H\alpha}(t)$	$S_H^{\alpha}(t)$
0.5	4.77388	1249.66
1	4.44208	677.701
1.5	4.2719	495.127
2	4.15991	402.719
2.5	4.07749	345.909
3	4.01278	306.982
3.5	3.9598	278.395
4	3.91512	256.365
4.5	3.8766	238.777
5	3.84283	224.351

The empirical results of the relative loss of Renyi and Harvrd & Charvat entropy are given in table 4. The results showed that the natural phenomenon holds for both the entropy measures, i.e., as the t increases the relative loss get decrease. While the results showed that, in all cases, the relative loss for Renyi and Harvrd & Charvat entropy are more than one by using a constant value of alpha=4. It is to be noted that by increasing the value of alpha, the relative loss also gets increased. The results conclude that these entropy measures are not good for the Pareto distribution.

Table 5: Relative loss of Awad et al and Arimoto's Entropy

t	θ=0.01 λ=0.03 α=2	
	$S_{\alpha A}(t)$	$S_{A\alpha}(t)$
0.5	-4.06492	-0.051439
1	-3.78909	-0.0873526
1.5	-3.64613	-0.120081
2	-3.55163	-0.150999
2.5	-3.4819	-0.180681
3	-3.42707	-0.209437
3.5	-3.38213	-0.237461
4	-3.34419	-0.264884
4.5	-3.31147	-0.291799
5	-3.28276	-0.318277

The empirical results of relative loss of Awad et al. and Arimoto's entropy are given in table 5. The results showed that the natural phenomenon holds for Awad et al entropy, while for Arimoto's it does not hold. While the results showed that, in all cases the relative loss for Awad et al. and Arimoto's entropy are negative, while an extra constant alpha=2 also used. Both entropy measures do not consider best for Pareto distribution Because of negative values of the relative loss of entropy measure.

Table 6: Relative loss of Awad et al and Arimoto's Entropy

t	θ=0.01 λ=0.03 α=3	
	$S_{\alpha A}(t)$	$S_{A\alpha}(t)$
0.5	-1.0168	-0.0248359
1	-0.946161	-0.0835247
1.5	-0.909919	-0.17166
2	-0.886067	-0.287335
2.5	-0.868511	-0.429341
3	-0.854728	-0.596812
3.5	-0.843444	-0.789074
4	-0.833927	-1.00558
4.5	-0.825723	-1.24589
5	-0.81853	-1.50959

The empirical results of the relative loss of Awad et al. and Arimoto's entropy are given in table 6. The results showed that the natural phenomenon holds for Awad et.al entropy, while for Arimoto's it does not hold. The results showed that, in all cases, the relative loss for Awad et.al and Arimoto's entropy are negative by using alpha=3. Again, these entropy measures lead to bad fit using Pareto distribution.

Table 7: Relative loss of Awad et al and Arimoto's Entropy

t	**θ=0.01 λ=0.03 α=4**	
	$S_{\alpha A}(t)$	**$S_{A\alpha}(t)$**
0.5	-0.735274	-0.0120431
1	-0.684171	-0.0809798
1.5	-0.65796	-0.249631
2	-0.640711	-0.557118
2.5	-0.628016	-1.04056
3	-0.618049	-1.73573
3.5	-0.609889	-2.67737
4	-0.603008	-3.89942
4.5	-0.597076	-5.43516
5	-0.591875	-7.3173

The empirical results of the relative loss of Awad et al. and Arimoto's entropy are given in table 7. The results showed that the natural phenomenon holds for Awad et.al entropy, while for Arimoto's it does not hold. The results showed that, in all cases, the relative loss for Awad et.al and Arimoto's entropy are negative with a constant parameter alpha=4. Both the entropy measures do not consider the best for the Pareto distribution because of negative values of the relative loss of entropy measures.

Graphical Representation

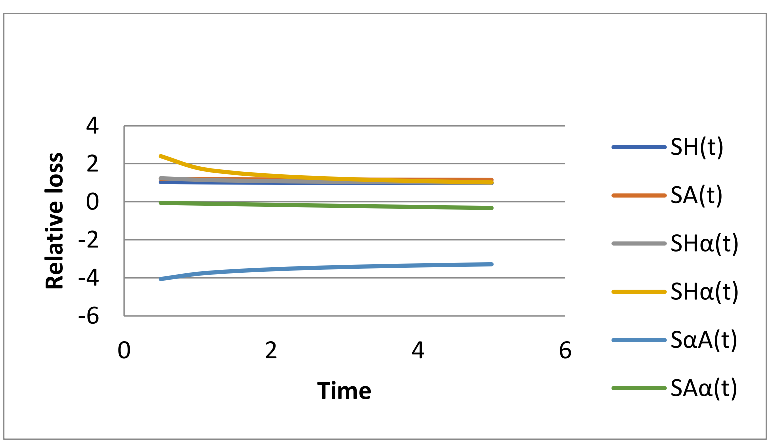

Figure 1: Relative loss of various Entropy Measures

The graphical representation of the relative loss of entropy measures is given in figure 1. Different lines are used for each entropy measure. Four lines are above zero and two lines are below zero, which means the relative loss for four entropy measures is positive and for two entropy measures it is negative, which are also shown in tables. The four entropy measures which are above zero give trends from maximum to minimum, i.e. as t increases the relative loss of four entropy measures gets decrease, which is the natural phenomenon. In rest of two entropy measures, the relative loss of one entropy measure gets decrease as t increases while the relative loss of second entropy measure gets increase as t increases. So, from overall, the study considers entropy measure with a blue line, i.e. Shannon entropy as best entropy, because it lies in between zero and one.

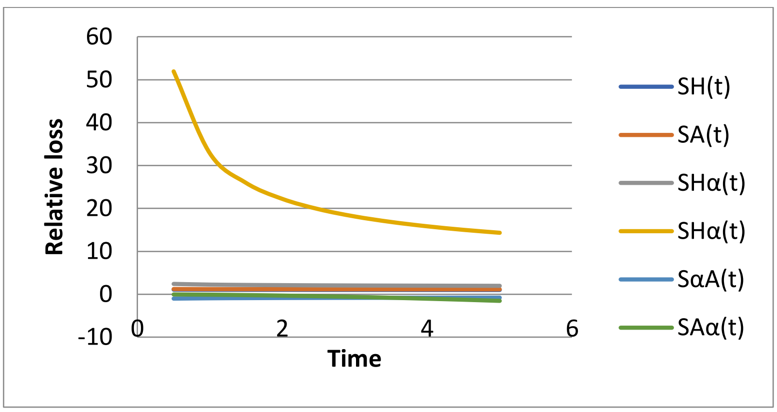

Figure 2: Relative loss of various Entropy Measures

The graphical representation of the relative loss of entropy measures are given in figure 2. Different lines are used for each entropy measure. Four lines are above zero and two lines are below zero, which means the relative loss for four entropy measures are positive and for two entropy measures it is negative, which are also shown in tables, after changing the value of alpha from 2 to 3. The four entropy measures which are above zero give trends from maximum to minimum, i.e., as t increases the relative of four entropy measures gets decrease, which is the natural phenomenon. In the rest of two entropy measures, the relative loss of one entropy measure gets decrease as t increases while the relative loss of second entropy measure get increase as t increases. So, from overall, the study considers entropy measure with the blue line, i.e., the Shannon entropy is the best entropy, because it lies in between zero and one.

CONCLUSION

In this study various entropy measures like Shannon, Renyi, Harvrd and Charvat, Awad, Awad et al and Arimoto's were derived for the Pareto and Truncated Pareto distribution. After the derivation of all entropy measure, the mathematical expression of the relative losses was derived for each entropy measure.

For estimation of the parameters, the procedure of maximum likelihood method was considered. Using a real data set in Statistical package "R" and find out the maximum likelihood estimate of Scale Parameter θ=0.01 and Shape parameter λ=0.03. The results showed that the natural phenomenon holds for both the entropy measures, i.e., as the t increases the relative loss get decrease. While the results showed that, in some cases, the relative loss for Shannon entropy is less than one, which is not the case in Awad entropy. So here, the Shannon entropy considers the best entropy measure for the Pareto distribution.

The empirical results of the relative loss of Renyi and Harvrd & Charvat entropy are given. The results showed that the natural phenomenon holds for both the entropy measures, i.e., as the t increases the relative loss get decreases. While the results showed that, in all cases the relative loss for Renyi and Harvrd & Charvat entropy is more than one, while different values of an extra constant alpha also used, which increased the values relative losses. So, comparing with Shannon entropy these two entropy measures are not best for Pareto distribution.

The empirical results of the relative loss of Awad et al. and Arimoto's entropy are given. The results showed that the natural phenomenon holds for Awad et al entropy, while for Arimoto's it does not hold. While the results showed that, in all cases the relative loss for Awad et al. and Arimoto's entropy are negative, while using different values of an extra constant alpha also used, which cannot remove the negative nature of the relative loss of entropy. Both entropy measures do not consider the best for Pareto distribution Because of negative values of the relative loss of entropy measure.

The results concluded that the Shannon entropy is the best entropy measures among others for the Pareto distribution.

This study considers various entropy measures for Pareto distribution and Truncated Pareto distribution, and also calculate the loss of entropy when underlying distribution is truncated Pareto distribution instead of Pareto distribution. The results of the relative loss of entropy measures showed that, the natural phenomenon holds in Shannon, Awad, Renyi and Harvrd & Charvat entropy measures for Pareto distribution, while in Awad et al. and Arimoto's it does not hold. Amongst the four entropy measures the Shannon entropy measure is considered best because it gives us minimum loss of information if one considers Truncated Pareto distribution instead of Pareto distribution.

RECOMMENDATIONS

The concept of this study is also incorporated in other life time distributions e.g. Log- logistics, Exponential, Weibull etc.

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APPENDIX 1

Detail Derivation of Entropy Measures for Pareto Distribution

Shannon Entropy of X

$$H(X) = -\int_{\theta}^{\infty} f(x;\theta,\lambda)\ \ln f(x;\theta,\lambda)dx$$

$$= -\left[\int_{\theta}^{\infty} \frac{\lambda\theta^{\lambda}}{x^{\lambda+1}}\ \ln \frac{\lambda\theta^{\lambda}}{x^{\lambda+1}}\, dx\right]$$

$$= -\left[\lambda\theta^{\lambda}\int_{\theta}^{\infty} \frac{1}{x^{\lambda+1}}\left\{\ln \lambda\theta^{\lambda} - \ln x^{\lambda+1}\right\} dx\right]$$

$$= -\left[\lambda\theta^{\lambda}\int_{\theta}^{\infty} x^{-\lambda-1}\left\{\ln \lambda\theta^{\lambda} - (\lambda+1)\ln x\right\} dx\right]$$

$$= -\left[\lambda\theta^{\lambda}\ln \lambda\theta^{\lambda}\int_{\theta}^{\infty} x^{-\lambda-1}\, dx - \lambda\theta^{\lambda}(\lambda+1)\int_{\theta}^{\infty} x^{-\lambda-1}\ln x dx\right]$$

$$= -\left[\lambda\theta^{\lambda}\ln \lambda\theta^{\lambda}\ \frac{x^{-\lambda}}{-\lambda} - \lambda\theta^{\lambda}(\lambda+1)\frac{x^{-\lambda}(1+\lambda\ln x)}{-\lambda^{2}}\right]$$

Putting limits to have

$$= -\left[\theta^{\lambda}\theta^{-\lambda}\ln \lambda\theta^{\lambda} - \frac{\theta^{\lambda}(\lambda+1)\theta^{-\lambda}(1+\lambda\ln\theta)}{\lambda}\right]$$

$$= -\left[\ln \lambda\theta^{\lambda} - \frac{(\lambda+1)(1+\lambda\ln\theta)}{\lambda}\right]$$

$$= -\left[\ln \lambda\theta^{\lambda} - \frac{(\lambda+\lambda^{2}\ln\theta+1+\lambda\ln\theta)}{\lambda}\right]$$

$$= -\left[\ln \lambda\theta^{\lambda} - \frac{\lambda}{\lambda} - \frac{\lambda^2 \ln\theta}{\lambda} - \frac{1}{\lambda} - \frac{\lambda \ln\theta}{\lambda} \right]$$

$$= -\left[\ln \lambda\theta^{\lambda} - 1 - \lambda \ln\theta - \frac{1}{\lambda} - \ln\theta \right]$$

$$= -\left[\ln \lambda + \lambda \ln\theta - 1 - \lambda \ln\theta - \frac{1}{\lambda} - \ln\theta \right]$$

$$= -\left[\ln \lambda - 1 - \frac{1}{\lambda} - \ln\theta \right]$$

$$= 1 + \frac{1}{\lambda} + \ln\theta - \ln\lambda$$

Other entropy measures are also derived in the similar way.

APPENDIX 2

Detail Derivation of Entropy Measures for Truncated Pareto Distribution

Shannon Entropy of Y

$$H(\mathrm{Y}) = -\int_{\theta}^{t} f(y;t;\theta,\lambda)\ \ln\ f(y;t;\theta,\lambda) dy$$

$$= -\int_{\theta}^{t} \frac{f(y;\theta,\lambda)}{F(t;\theta,\lambda)} \ln \frac{f(y;\theta,\lambda)}{F(t;\theta,\lambda)} dy$$

$$= -\frac{1}{F(t;\theta,\lambda)} \left[\int_{\theta}^{t} \frac{\lambda\theta^{\lambda}}{y^{\lambda+1}} \ln \frac{\lambda\theta^{\lambda}}{y^{\lambda+1}} \Big/ F(t;\theta,\lambda)\, dy \right]$$

$$= -\frac{1}{F(t;\theta,\lambda)} \left[\lambda\theta^{\lambda} \int_{\theta}^{t} y^{-\lambda-1} \left\{ \ln \lambda\theta^{\lambda} - \ln y^{\lambda+1} - \ln F(t;\theta,\lambda) \right\} dy \right]$$

$$= -\frac{1}{F(t;\theta,\lambda)} \left[\lambda\theta^{\lambda} \int_{\theta}^{t} y^{-\lambda-1} \left\{ \ln \lambda\theta^{\lambda} - (\lambda+1) \ln y - \ln F(t;\theta,\lambda) \right\} dy \right]$$

$$= -\frac{1}{F(t;\theta,\lambda)}\begin{bmatrix} \lambda\theta^{\lambda}\ln\lambda\theta^{\lambda}\int\limits_{\theta}^{t} y^{-\lambda-1}\,dy - \lambda\theta^{\lambda}\left(\lambda+1\right)\int\limits_{\theta}^{t} y^{-\lambda-1}\ \ln y dy - \\ \lambda\theta^{\lambda}\ln F(t;\theta,\lambda)\int\limits_{\theta}^{t} y^{-\lambda-1}dy \end{bmatrix}$$

$$= -\frac{1}{F(t;\theta,\lambda)}\begin{bmatrix} \left\{\lambda\theta^{\lambda}\ln\lambda\theta^{\lambda}\,\frac{y^{-\lambda}}{-\lambda}\right\} - \left\{\lambda\theta^{\lambda}\left(\lambda+1\right)\frac{y^{-\lambda}\left(1+\lambda\ln y\right)}{-\lambda^{2}}\right\} - \\ \left\{\lambda\theta^{\lambda}\ln F(t;\theta,\lambda)\,\frac{y^{-\lambda}}{-\lambda}\right\} \end{bmatrix}$$

Putting limits to have

$$= -\frac{1}{F(t;\theta,\lambda)}\begin{bmatrix} \left\{-\theta^{\lambda}\ln\lambda\theta^{\lambda}\left(t^{-\lambda}-\theta^{-\lambda}\right)\right\} + \left\{\frac{\theta^{\lambda}\left(\lambda+1\right)}{\lambda}\begin{Bmatrix} t^{-\lambda}\left(1+\lambda\,\mathrm{lnt}\right) - \\ \theta^{-\lambda}\left(1+\lambda\ln\theta\right) \end{Bmatrix}\right\} - \\ \left\{-\theta^{\lambda}\ln F(t;\theta,\lambda)\left(t^{-\lambda}-\theta^{-\lambda}\right)\right\} \end{bmatrix}$$

$$= -\frac{1}{F(t;\theta,\lambda)}\begin{bmatrix} \left\{\theta^{\lambda}\theta^{-\lambda}\ln\lambda\theta^{\lambda} - \theta^{\lambda}t^{-\lambda}\ln\lambda\theta^{\lambda}\right\} + \\ \left\{\frac{\left(\lambda\theta^{\lambda}+\theta^{\lambda}\right)}{\lambda}\left\{t^{-\lambda}+\lambda t^{-\lambda}\,\mathrm{lnt} - \theta^{-\lambda} - \lambda\theta^{-\lambda}\ln\theta\right\}\right\} - \\ \left\{\left(\theta^{\lambda}\theta^{-\lambda} - \theta^{\lambda}t^{-\lambda}\right)\ln F(t;\theta,\lambda)\right\} \end{bmatrix}$$

$$= -\frac{1}{F(t;\theta,\lambda)}\begin{bmatrix} \left\{\ln\lambda\theta^{\lambda} - \theta^{\lambda}t^{-\lambda}\ln\lambda\theta^{\lambda}\right\} + \left\{\frac{\left(\lambda\theta^{\lambda}+\theta^{\lambda}\right)}{\lambda}\begin{Bmatrix} t^{-\lambda}+\lambda t^{-\lambda}\,\mathrm{lnt} - \\ \theta^{-\lambda} - \lambda\theta^{-\lambda}\ln\theta \end{Bmatrix}\right\} \\ -\left\{\left(1-\theta^{\lambda}t^{-\lambda}\right)\ln F(t;\theta,\lambda)\right\} \end{bmatrix}$$

$$= \frac{1}{F(t;\theta,\lambda)}\begin{bmatrix} \left\{\left(1-\theta^{\lambda}t^{-\lambda}\right)\ln F(t;\theta,\lambda)\right\} - \left\{\ln\lambda\theta^{\lambda} - \theta^{\lambda}t^{-\lambda}\ln\lambda\theta^{\lambda}\right\} - \\ \left\{\frac{\left(\lambda\theta^{\lambda}+\theta^{\lambda}\right)}{\lambda}\left\{t^{-\lambda}+\lambda t^{-\lambda}\,\mathrm{lnt} - \theta^{-\lambda} - \lambda\theta^{-\lambda}\ln\theta\right\}\right\} \end{bmatrix}$$

OR

$$= \frac{t^{\lambda}}{t^{\lambda}-\theta^{\lambda}}\left[\begin{array}{l}\left\{\left(1-\theta^{\lambda}t^{-\lambda}\right)\ln\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)\right\}-\left\{\ln\lambda\theta^{\lambda}-\theta^{\lambda}t^{-\lambda}\ln\lambda\theta^{\lambda}\right\}- \\ \left\{\frac{\left(\lambda\theta^{\lambda}+\theta^{\lambda}\right)}{\lambda}\left\{t^{-\lambda}+\lambda t^{-\lambda}\ln t-\theta^{-\lambda}-\lambda\theta^{-\lambda}\ln\theta\right\}\right\}\end{array}\right]$$

Other entropy measures are also derived in the same way.

APPENDIX 3

Detail Derivation of Relative loss of Entropy Measures when using Y instead of X

Relative loss of Shannon Entropy

$$S_H(t)=\frac{\mathrm{H}(X)-H(Y)}{\mathrm{H}(X)}$$

OR

$$=1-\left[\frac{\frac{t^{\lambda}}{t^{\lambda}-\theta^{\lambda}}\left[\begin{array}{l}\left\{\left(1-\theta^{\lambda}t^{-\lambda}\right)\ln\left(\frac{t^{\lambda}-\theta^{\lambda}}{t^{\lambda}}\right)\right\}-\left\{\ln\lambda\theta^{\lambda}-\theta^{\lambda}t^{-\lambda}\ln\lambda\theta^{\lambda}\right\}- \\ \left\{\frac{\left(\lambda\theta^{\lambda}+\theta^{\lambda}\right)}{\lambda}\left\{t^{-\lambda}+\lambda t^{-\lambda}\ln t-\theta^{-\lambda}-\lambda\theta^{-\lambda}\ln\theta\right\}\right\}\end{array}\right]}{1+\frac{1}{\lambda}+\ln\theta-\ln\lambda}\right]$$